Société de Calcul Mathématique SA

Outils d'aide à la décision

Archimedes' Weighing Method:

Improvements and Modern Applications

by Bernard Beauzamy

June 2013

Introduction

As it is well-known (see for instance our recent book [AMW]), Archimedes first established his comparison between the volume of a sphere and the volume of the cylinder which contains it, using a "Weighing Method". Then, he turned to "analytic" methods, and he introduced what became later the integral and differential calculus. Using such analytic methods, he proved for instance that the surface of a sphere is 4 times the surface of its great circle.

We are going to show that this result may be easily obtained using the Weighing Method. In some sense, we move backwards, using the methods first introduced by Archimedes. We start with surfaces and then consider volumes.

We then give modern applications of such "Weighing Methods".

Part I : Surfaces

We start with a lemma due to Archimedes, which reduces the computation of the area of a circle to the computation of the area of a triangle.

Lemma 1 (Archimedes : "On the measure of the circle", Proposition 1).

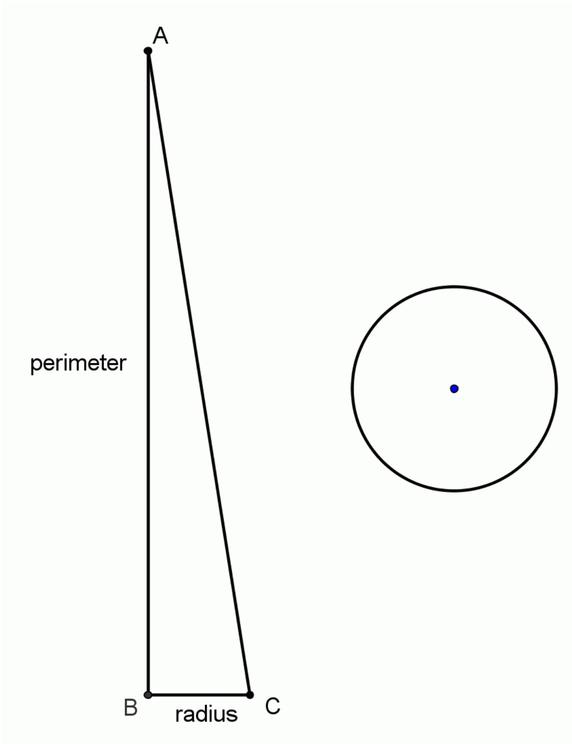
Let a circle with radius r *and let* ABC *be a triangle with right angle at* B*. If*

$$AB = perimeter(circle)$$

and

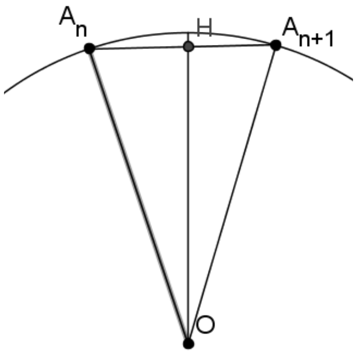
$$BC = radius(circle),$$

then the triangle and the circle have the same area.



Proof of Lemma 1

We inscribe in the circle a regular polygon with summits A_n. The area of a small triangle is $\frac{A_nA_{n+1} \times OH}{2}$ and the area of the polygon is $\frac{\sum_n A_nA_{n+1} \times OH}{2}$



When $n \to +\infty$, $a\left(polygon\right) \to a(circle)$,

$\sum_n A_nA_{n+1} \to l\left(circle\right)$, $OH \to r$

where $a\left(polygon\right)$ denotes the area of the polygon, $l\left(circle\right)$ is its perimeter, and r is the radius. So we find :

$$a\left(circle\right) = \frac{l(circle) \times r}{2}$$

which proves the lemma.

Corollary 2. - *For any* r,

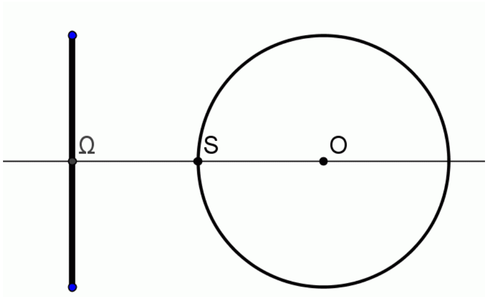
$$\frac{A_r}{r^2} = \frac{l_r}{2r}$$

where A_r is the area of the circle of radius r and l_r its perimeter.

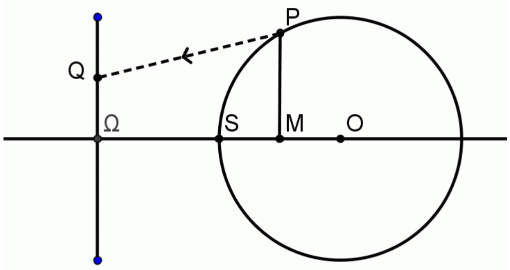
This formula is an obvious consequence of Lemma 1 (since $A_r = \frac{rl_r}{2}$), but it is interesting in itself. The Greeks knew that the quotient of the perimeter of a circle by its diameter is constant (they did not use the notation π) ; they knew also that the quotient of the area of a circle by the square of its radius is also constant, and the above formula shows that both definitions of π coincide.

*****

In what follows, we consider a sphere, infinitely thin (all the mass is at the surface), with radius r. We will study its equilibrium with a disk (also infinitely thin), in the following situation:



The sphere has its center at O and has radius r, and $SO = S\Omega = r$. The disk is represented by its vertical cut (infinitely thin), so it faces the sphere. The bar of the balance is the straight line passing through Ω, S, O and S is the fixed point.



We first consider the left half-sphere with summit at S and basis at O, and the following transformation φ from the half sphere to the disk with center Ω :

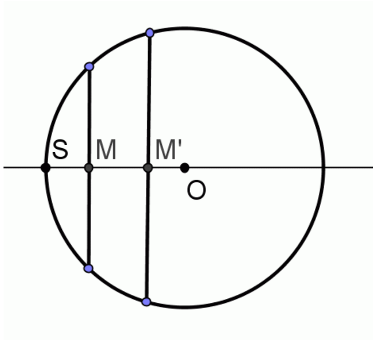
$$P \rightarrow Q = \varphi(P)$$

where $\Omega Q = SM$, M being the projection of P on the axis. The points P, M, Ω, Q belong to the same plane.

Proposition 3. – *The transformation* φ *from the left half-sphere onto the disk preserves the moment with respect to the point* S.

This means that, for any subset E of the half-sphere, the moment of E with respect to S is equal to the moment of $\varphi(E)$ with respect to the same point. Or, in other words, E and $\varphi(E)$ are in equilibrium with respect to S.

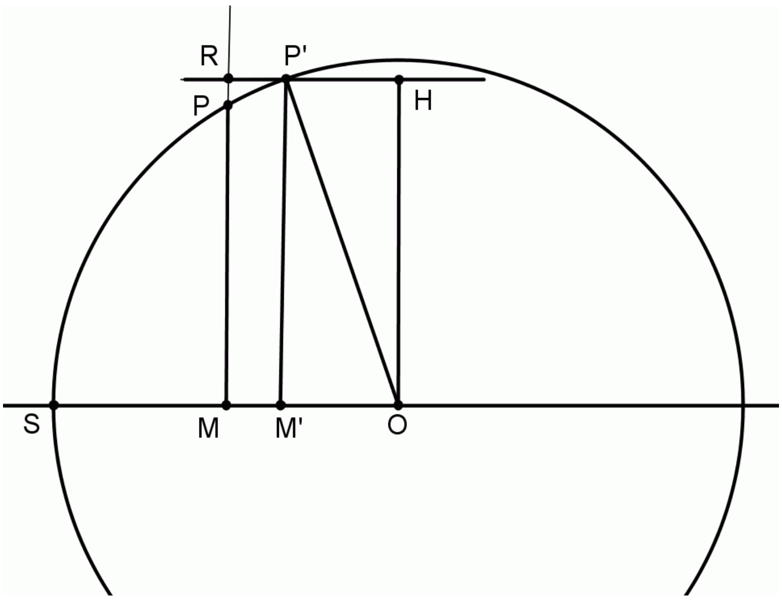
Proof of Proposition 3



Obviously, it is enough to prove this proposition for spherical caps.

A spherical cap $SC(x, x')$ is the portion of the sphere between the two parallel planes, orthogonal to the axis $SO,$ passing through $M,$ such that $SM = x,$ and $M',$ such that $SM' = x'.$

Let us consider a thin slice of sphere, spherical cap between the points M and M'. We set $SM = x$ and $MM' = \tau$. This slice has its center of gravity on the segment MM'.



Let us consider the segment PP' ; the triangles $PP'R$ and OHP' are similar (orthogonal segments), and therefore:

$$\frac{OH}{OP'} = \frac{P'R}{PP'}$$

which gives:

$$PP' = P'R \times \frac{OP'}{OH} \simeq \frac{r\,\tau}{\sqrt{r^2 - (r-x)^2}}$$

Therefore, the moment with respect to S of the slice, spherical cap $SC(x, x')$ between M and M' is:

$$M_S(slice) \simeq mass(PP') \times perimeter(circle) \times SM$$

which gives:

$$M_S(slice) \simeq \frac{r\tau}{\sqrt{r^2 - (r-x)^2}} \times 2\pi\sqrt{r^2 - (r-x)^2} \times x = 2\pi r x \tau$$

After the transformation φ, the points P, P' become points Q, Q' with $\Omega Q = x$ and $QQ' = \tau$. Therefore, the moment with respect to S of the annulus between Q and Q' is:

$$M_S(annulus) = a(annulus) \times \text{distance}(\Omega S) = 2\pi x\tau \times r$$

and both moments are equivalent when $\tau \to 0$, which proves Proposition 3.

Remark. – This proposition really deals with surfaces. It is not true that the circle generated by P (to which we an assign a weight, proportional to its perimeter) is in equilibrium with the circle generated by Q.

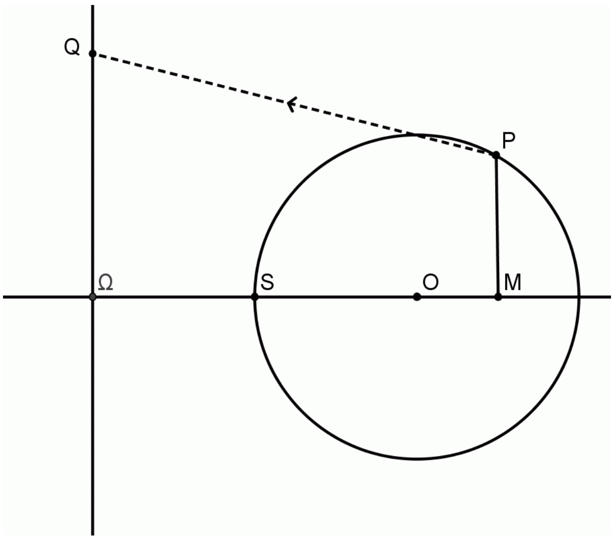
Corollary 4. – *The half-sphere is in equilibrium with the disk with same radius.*

Indeed, we apply the Proposition to all x, $0 \le x \le r$.

Corollary 5. – *For any ρ, ρ' $0 \le \rho \le \rho' \le r$, the spherical cap $SC(\rho, \rho')$ is in equilibrium with the annulus $Ann(\Omega; \rho, \rho')$ (annulus with center Ω, between the two radii ρ and ρ').*

Here, we apply the Proposition only to the values between ρ and ρ'.

We now turn to the right half-sphere. The transformation φ is the same : $\Omega Q = SM$, but this time it sends the right half-sphere to the annulus $Ann(\Omega; r, 2r)$. The momentum of a slice between x and $x+\tau$ (with $x \ge r$) is, as before:



$$M_S(slice) = 2\pi r x\tau$$

and so is the moment of the corresponding annulus in the disk. So we obtain:

Proposition 6. – *The transformation φ from the right half-sphere onto the annulus $Ann(\Omega; r, 2r)$ preserves the moment with respect to the point S.*

Corollary 7. – *The right half sphere is in equilibrium with the annulus $Ann(\Omega; r, 2r)$.*

Corollary 8. – *The whole sphere is in equilibrium, with respect to* S*, with the disk* $D(\Omega, 2r)$.

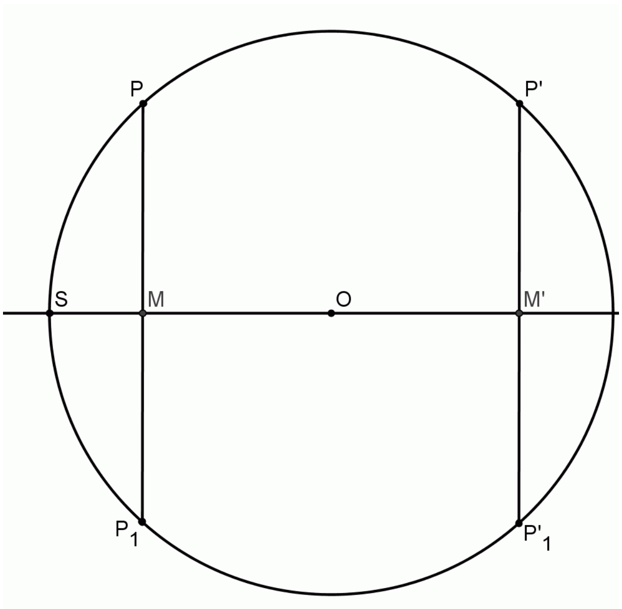
But the center of gravity of the whole sphere is at the point O. Since $SO = S\Omega$, we deduce that the weights of both solids are identical, which means that their areas are equal:

Corollary 9. – *The area of the sphere of radius* r *is equal to the area of the disk of radius* $2r$.

Since the area of a disk of radius $2r$ is 4 times the area of a disk of radius r, we get:

Theorem (Archimedes). – *The area of the sphere is equal to four times the area of a great circle.*

We can also apply the above results to spherical caps. Let $SC(0, x)$ be any spherical cap with summit at S and base at M, such that $SM = x$, and let G be its center of gravity (obviously on the axis). Let $SC(2r - x, 2r)$ be the "opposite" spherical cap (symmetric of the previous one, with respect to the center of the sphere). Both caps have the same area; let G' be the center of gravity of the second cap. The center of gravity of the two caps together is O.



By Corollaries 5 and 7, the set made of both caps is in equilibrium with the disk $D(\Omega, x)$ plus the annulus $Ann(2r - x, 2r)$. Since $\Omega S = SO$, the set "two caps" and the set "disk plus annulus" have the same areas.

But the total area of "disk plus annulus" is $\pi x^2 + \pi(2r)^2 - \pi(2r - x)^2 = 4\pi rx$, and therefore the area of the cap $SC(0, x)$ is $2\pi rx$.

So we have obtained:

Theorem (Archimedes). – *For any spherical cap of height* x*, built on a sphere with radius* r*, the area of the cap is* $a = 2\pi rx$.

We also deduce the position of the center of gravity of the cap. The equilibrium shows that:

$$a\left(D\left(\Omega,x\right)\right)\times r = a\left(SC\left(0,x\right)\right)\times SG$$

which gives:

$$SG = \frac{x}{2}$$

So we have proved:

Theorem (Archimedes). – *The center of gravity of a cap of height* x *is on the axis of the cap, at distance* $\frac{x}{2}$ *from the summit.*

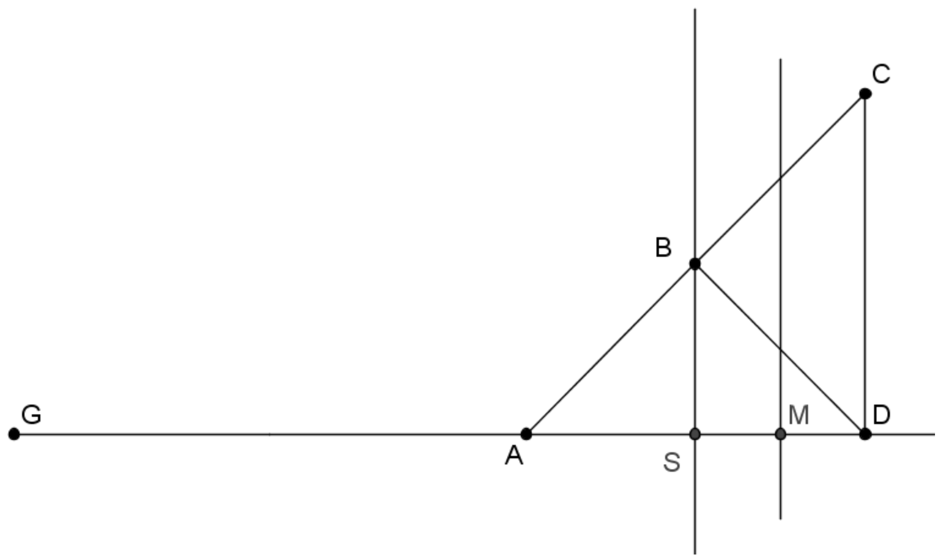
Part II : Volumes

We now turn to the investigation of volumes and position of centers of gravity.We start with a cone :

Theorem 1. – *The center of gravity of a cone is on the axis, at 3/4 starting from the summit.*

Proof of Theorem 1

We consider a cone of summit A and base $CD = 2$ (only a half-cone is represented on the figure) ; the height is $AD = 2$; so the cone has a right angle at the summit. We also have $SD = AS = 1.$



We consider the piece of cone generated by the rotation of the triangle BCD around the x axis. The moment with respect to S of the slice at M is (with $SM = x$):

$$moment_S\left(slice\right) = x\pi\left(\left(x+1\right)^2 - \left(x-1\right)^2\right) = 4\pi x^2$$

Let G be the point, left of $S,$ such as $SG = 4.$ We may write the above formula:

$$moment_S\left(slice\right) = SG \times \pi x^2$$

When M moves between S and D, that is when $0 \le x \le 1,$ the slice generates the difference between the truncated cone $SBCD$ and the cone BSD; the term $SG \times \pi x^2$ generates the moment of a cone with base of radius $r = 1,$ height $h = 1,$ and center of gravity at G .

So, the difference between the truncated cone $SBCD$ and the cone BSD is in equilibrium around S with the cone C_1 of base of radius $r = 1$, height $h = 1$, and center of gravity at G.

To remove the cone BSD from the right is the same as to add ASB on the left, which gives the complete cone C_2 with summit A, base CD of radius 2 and height $AD = 2$.

Since the height and radius are double of that of C_1, we have:

$$vol(C_2) = 8vol(C_1)$$

Let X be the unknown position of the center of gravity of C_2 (it must be on the axis). The above equilibrium gives:

$$SG \times vol(C_1) = SX \times vol(C_2)$$

that is:

$$SX = \frac{SG}{8}$$

Since $SG = 4$, we find $SX = \frac{1}{2}$ and $AX = \frac{3}{2} = \frac{3AD}{4}$, which shows that the center of the gravity is at 3/4 of the height, starting at the summit.

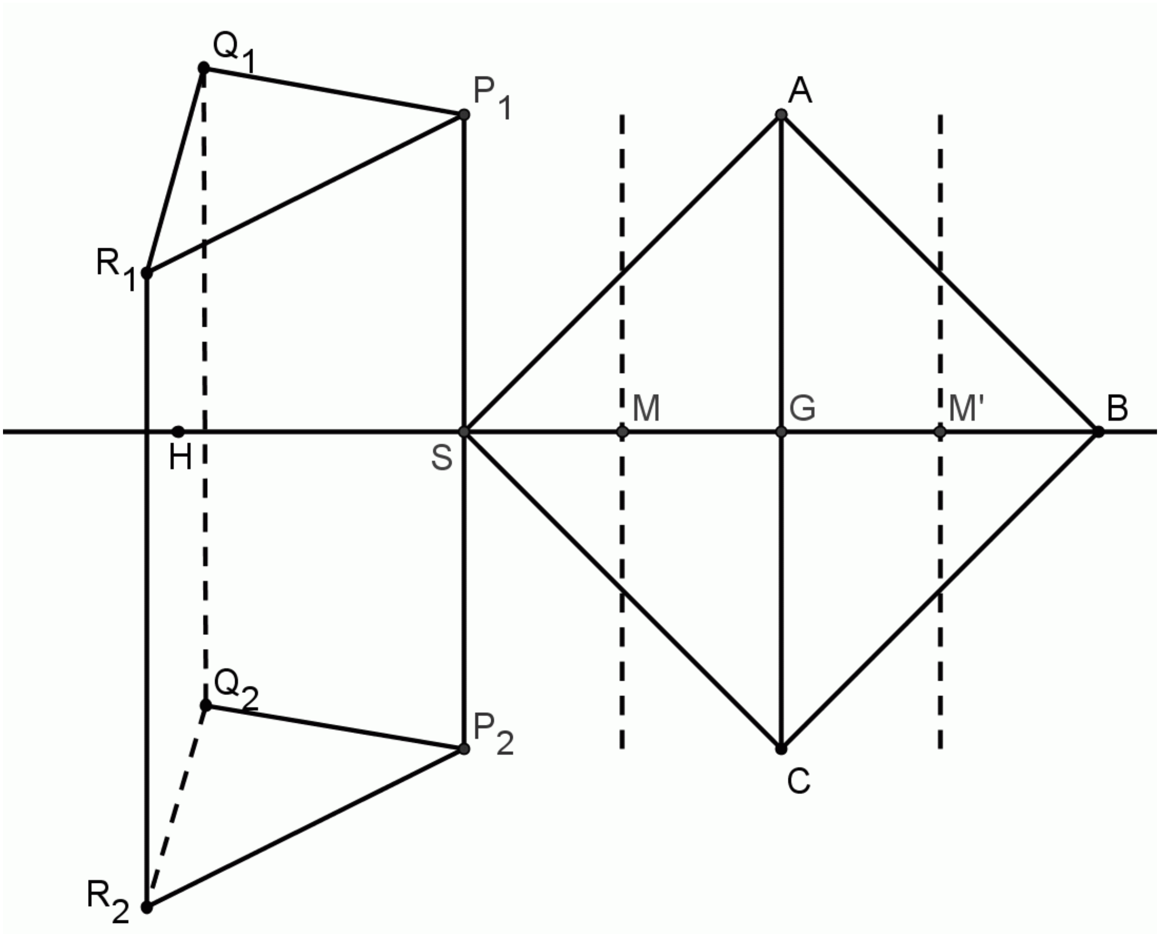
This result, as it stands, was obtained only for the particular cone above, which has a right angle. But, quite clearly, it extends to any cone of revolution, since the moments with respect to the summit are proportional to the square of the base: increasing the radius of the cone increases the same way all moments, and the center of gravity is not changed. This proves the Theorem.

We deduce a formula for the volume of a cone.

Theorem 2. – *The volume of a cone with right angle and base of radius* r *is* $V = \frac{\pi r^3}{3}$

Proof of Theorem 2

We consider the following two objects:



First, a juxtaposition of two opposite cones; in this picture, $GS = GA = GB = GC = r$.

Second, a prism, $P_1Q_1R_1P_2Q_2R_2$ with right angle at P_1 and P_2, with height $SH = r$ and base $R_1Q_1 = R_2Q_2 = r$.

We claim that these two objects are in equilibrium with respect to S.

Let us compute the moment with respect to S of the two slices of the cones, one at M such that $SM = x$ and one at M' such that $SM' = 2r - x$. We have:

$$Moment_S\left(slice_M \cup slice_{M'}\right) = x \times \pi x^2 + \left(2r - x\right) \times \pi x^2 = 2\pi r x^2$$

Let us write :

$$2\pi r x^2 = x \times 2\pi r \times x$$

This is the moment (with respect to S) of a rectangular slice at distance x from S, with height $2\pi r$ and width x. Since the height is constant and the width depends linearly on x, this rectangular slice generates a prism, namely $P_1Q_1R_1P_2Q_2R_2$.

So these two objects are in equilibrium. Let G_P be the center of gravity of the prism. Since the center of gravity of the triangles $Q_1P_1R_1$ and $Q_2P_2R_2$ are on the median, at 2/3 starting from the summit, this is true for the whole prism, so $SG_P = \frac{2SH}{3}$.

The equilibrium implies :

$$SG_P \times vol(prism) = SG \times vol(2\ cones)$$

But the volume of the prism is:

$$vol(prism) = a(base) \times height = \frac{r^2}{2} \times 2\pi r = \pi r^3$$

and $SG_P = \frac{2r}{3}$. Since $SG = r$ and $vol(2\ cones) = 2vol(cone)$, we get:

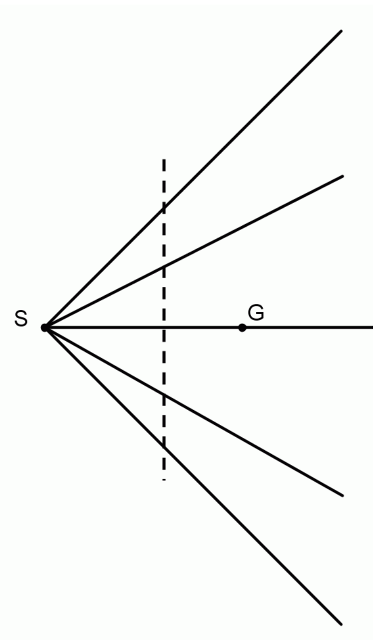
$$vol(cone) = \frac{2r}{3}\frac{\pi r^3}{2r} = \frac{\pi r^3}{3}$$

which proves the Theorem.

Corollary 3. - *The volume of any cone of revolution, of height* h *and radius* r *is* $\frac{\pi h r^2}{3}$

Proof of Corollary 3

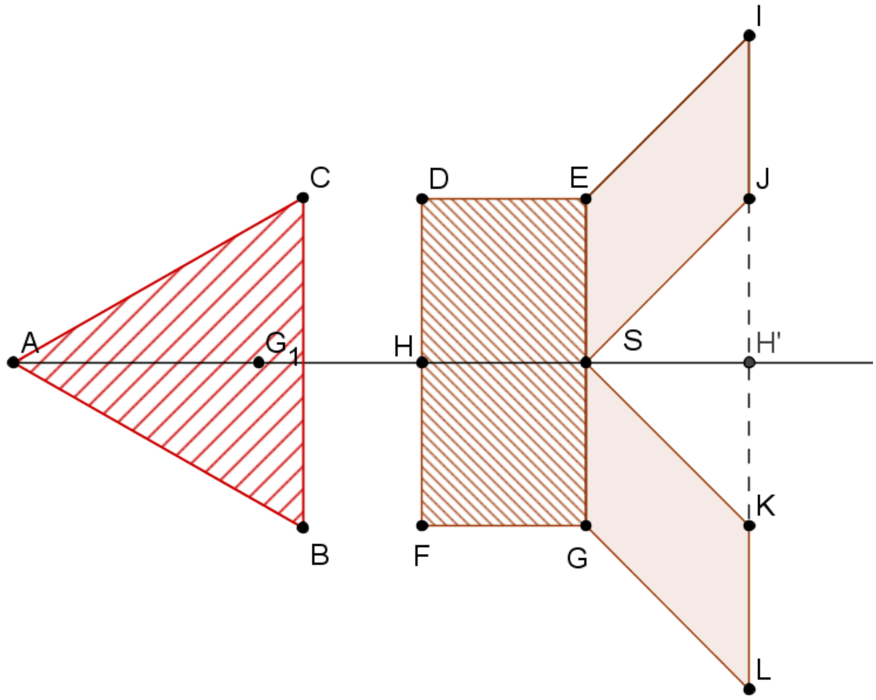
We start with a cone with right angle, with height and radius h; its volume is $\frac{\pi h^3}{3}$, by Theorem 2. Now, consider a cone with radius $r = \lambda h$ (see figure). The center of gravity is the same. The moment with respect to S is, for a slice at distance x from S for the right angle cone, $Moment_S(slice) = \pi x^2$, and, for the second cone, $\pi \lambda^2 x^2$. Since the centers of gravity have not changed, this means that the volume has been multiplied by $\lambda^2 = \frac{r^2}{h^2}$; this proves the corollary.



Of course, the quantity $\pi r^2 h$ is the volume of a cylinder of radius r and height h, so we get:

Corollary 4 (Archimedes). – *The volume of the cone is one third of the volume of the cylinder with same base and same height.*

This statement may be obtained directly by weighing techniques; see our book [AMW], but it requires to combine several cones together, as we now see:



In this picture, the cone ABC, with summit at A, height r, base BC of radius r, plus the cylinder $DEFG$, height $HS = r$, radius $HD = HF = r$, is in equilibrium around S with the piece of cone generated by the rotation of $SEIJ$: this is the cone IHL, with summit at H, truncated at E, minus the cone JSK. We have $ES = IJ = SH' = r$.

In order to see this, we use the fact that the moment, with respect to S, of a slice of the right set is:

$$M_1 = x\pi\left(x+r\right)^2 - x\pi x^2 = 2\pi r x^2 + \pi x r^2$$

The term $2r \times \pi x^2$ corresponds to the moment of the slice of a cone, with its center of gravity at the point G_1, with $SG_1 = 2r$.

The term $x \times \pi r^2$ corresponds to the moment of the slice of a cylinder, with radius r and height r : this is the cylinder $DEFG$.

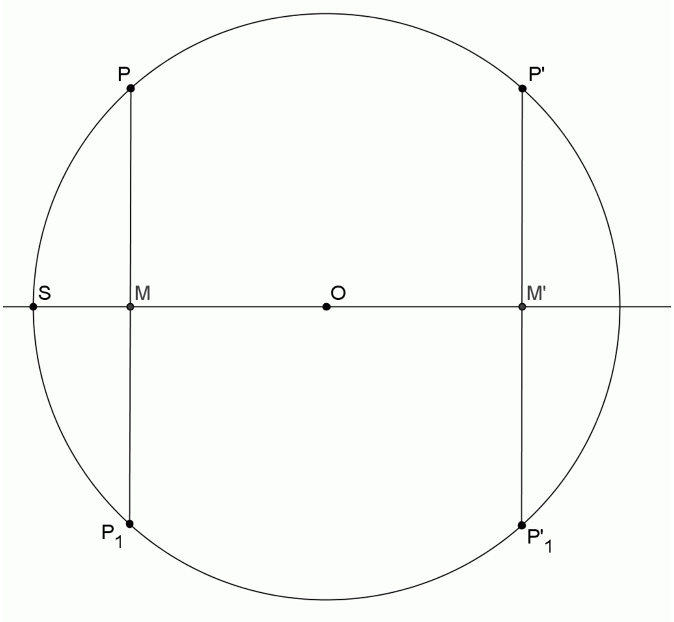
So the truncated cone $EIGL$, minus the cone JSK is in equilibrium around S with the cone ABC, center of gravity at G_1, plus the cylinder $DEFG$. The result follows, using the same computations as previously.

*****

The sphere and the cylinder

We now come back on the result Archimedes originally proved using his Weighing Method.

Instead of considering the moment of one bar, as Archimedes did, we now consider the moment with respect to S of two bars, symmetric with respect to the center O of the sphere ; this improves slightly upon Archimedes' approach:



The moment with respect to S of the slice at M is:

$$M_S\left(slice_M\right) = SM \times \pi MP^2$$

and for the slice at M':

$$M_S\left(slice_{M'}\right) = SM' \times \pi MP^2$$

But since $SM + SM' = 2SO = 2r,$ we get:

$$M_S\left(slice_M \cup slice_{M'}\right) = 2\pi r MP^2$$

But:

$$MP^2 = OP^2 - OM^2 = \left(OP - OM\right)\left(OP + OM\right) = \left(OS - OM\right)\left(OS + OM\right) = SM \times SM'$$

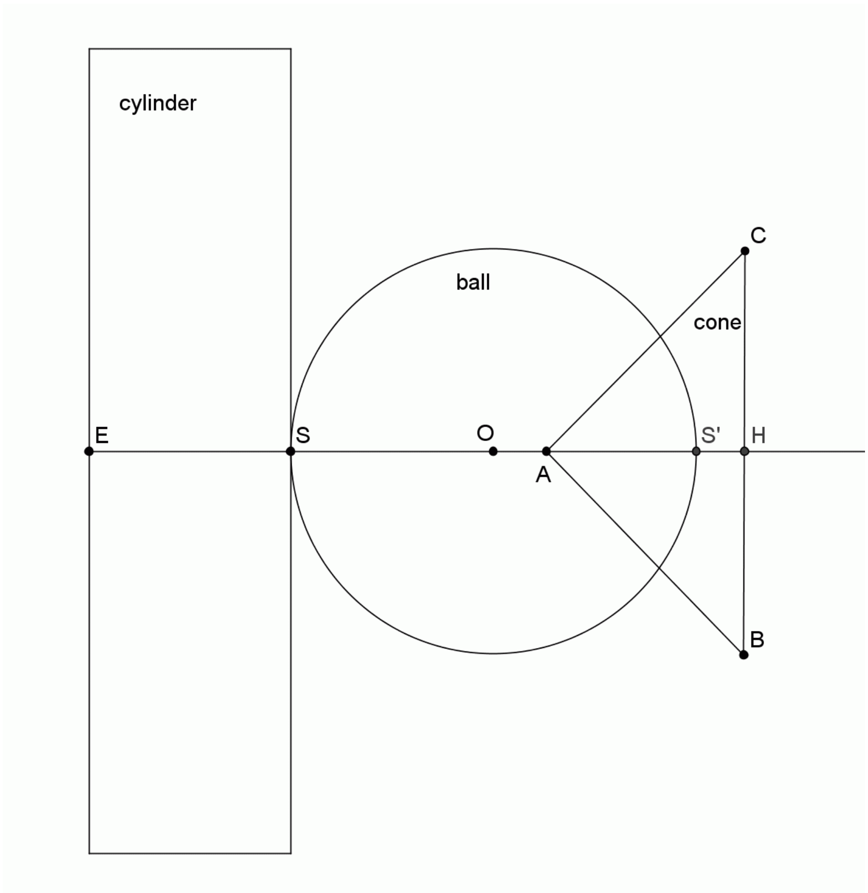
Let, as before, $SM = x$; we get:

$$M_S\left(slice_M \cup slice_{M'}\right) = 2\pi rx\left(2r - x\right) = 4\pi r^2 x - 2\pi r x^2$$

It follows from this formula that the two slices (considered together) are in equilibrium with respect to S with the following two objects (considered together) :

- At distance $x,$ left of $S,$ a disk of radius $2r$;
- At distance $2r,$ right of $S,$ a disk of radius $r.$

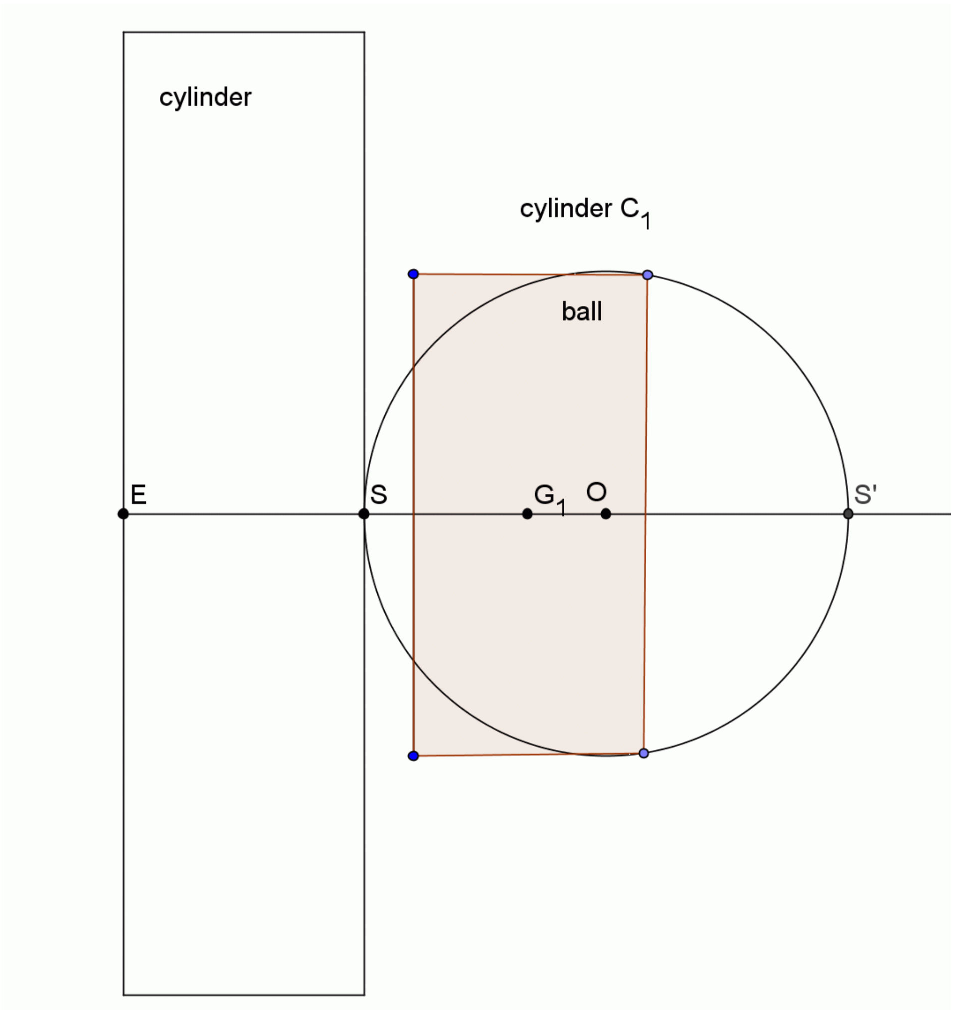
When M moves between S and O, the first disk generates a cylinder, since the radius is constant. This cylinder has a height equal to r .



The second disk generates a cone, since the radius increases with the distance. This cone has its center of gravity at S', point symmetric to S with respect to O, (so $SS' = 2r$).

The ball, where it is, is in equilibrium with respect to S with the cylinder of height r, radius $2r$, put left of S, and the cone of height r, circular base of radius r, with its center of gravity at S'.

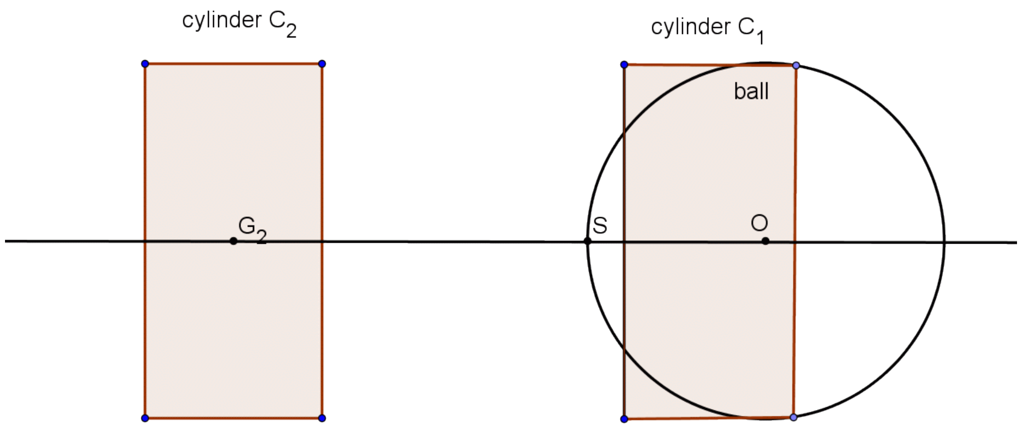
Since a cone has its center of gravity on the axis, at 3/4 from the summit, we may take a cone which has its summit at A with $OA = r/4$. Of course, any rotation of the cone around S' will have the same weight and moment with respect to S.



Now, still keeping the equilibrium, we want to replace the cone ABC by a cylinder C_1, with same base BC (radius r) and same height $AH = r$. Since the volume of the cylinder is triple of the volume of the cone (Corollary 4 above), the center of gravity G_1 will satisfy:

$$SG_1 = \frac{1}{3}SS' = \frac{2r}{3}$$

Let us decrease the radius of the left cylinder from $2r$ to r, still keeping the equilibrium and the same height, namely r. Then the weight is divided by 4 , so the center of gravity of this new cylinder C_2 will satisfy $SG_2 = \frac{4r}{2} = 2r$ and the height is $4r$:



On this picture, the red cylinder C_2 (left) is in equilibrium around S with the ball and the red cylinder C_1 (right).

This means:

$$2r\, vol(C_2) = r\, vol(ball) + \frac{2r}{3} vol(C_1)$$

But since C_1 and C_2 have the same volume :

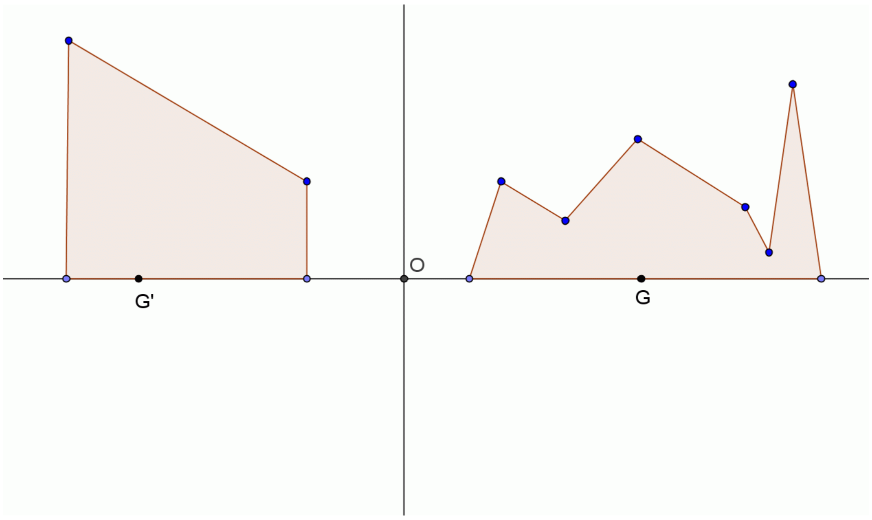
$$vol(Ball) = \frac{3}{4} vol(C_1)$$

The cylinder C_1 has radius r and height r, so its height is one half of the height of the cylinder enclosing the ball. We get finally:

Theorem (Archimedes). – *The volume of the ball is* $\frac{2}{3}$ *of the volume of the enclosing cylinder (same radius, height* $2r$*).*

A simple application

We have a solid of revolution, which is irregular (right of the figure below), and we want to determine a more regular solid, with same volume and same moment with respect to a given point. On the left of the picture below, the more regular solid is "decreasing", from left to right. Only the upper part of the solids appear on the picture below, but they are assumed to be revolution solids, with respect to the Ox axis.



We discretize our original solid in small cylinders ; it does not matter whether these cylinders have equal thickness or not). Then we can put these cylinders in decreasing order of radii on the left side of the axis : we clearly have a solid with same volume, which is decreasing in the sense we explained.

Now, if we move this new solid to the left, the moment increases, and if we move it towards the origin, the moment decreases. So there will be a unique position where the old solid and the new one will be in equilibrium.

The position of the center of gravity G' is very easy to characterize : indeed, one must have $OG = OG'$, since the weights of the two solids are equal.

Part III: Modern Times

A general use of Archimedes Weighing Method (AWM) is the following: we have some "signal" (this word being taken in a very general sense) which is unknown, or not completely known, and we generate artificially another "signal" and we compare it to the original one.

An example (see [AMW]) is given by the receiver in a GPS system: each satellite sends a signal, from which it is possible to deduce the distance from the receiver to the satellite. But this signal is usually too weak to be exploited directly. So the receiver generates a similar signal, and tries time translations of the generated signal, and finds the time translated signal which fits best with the received one. This is a typical application of a Weighing Method.

Such applications are always "robust", because the signal which is emitted is well-known; this is why they are used in difficult situations.

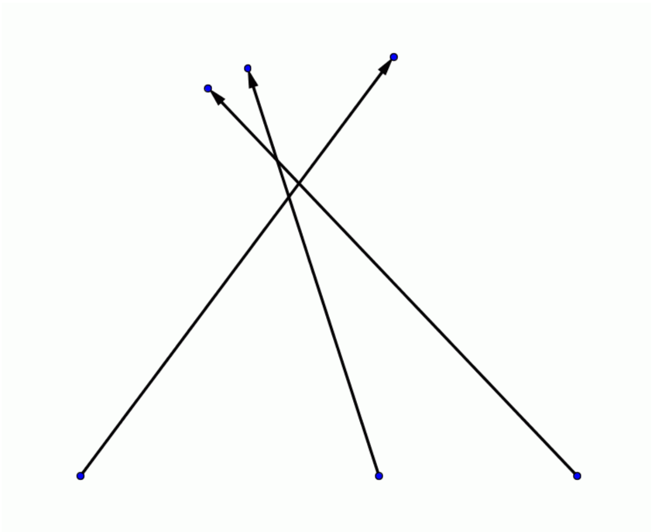
A similar idea was used by SCM in defining "prospective indicators" (see http://scmsa.eu/fiches/SCM_Indicateurs_prospectifs.pdf):

We study some quantity I, say the price of corn, and we want to anticipate its variation over a one year period. We find in some database, such as Bloomberg, a set of 3 indicators I_1, I_2, I_3, such that the triple so defined is best correlated to our indicator, with a one year time shift. Say for instance that, with some high probability, I increases when I_1 increases and both I_2, I_3 decrease. Then we can say that the information we require (namely the variation of I) is almost the same as the information generated by the triple I_1, I_2, I_3. We generate an artificial information, namely I_1, I_2, I_3, and we use it in order to predict I, since they are almost "in equilibrium".

A more general situation occurs when the information we generate artificially is also subject to some uncertainty, or if we need to reconcile several artificially generated informations, which do not seem to be compatible at first sight.

A typical example is that of GPS, when more than 3 satellites are seen: each satellite sends its own information of distance, which means that the receiver is situated on some sphere centered at the satellite. Usually, 4 spheres or more have an empty intersection, so probabilistic tools are needed to evaluate, for each position of the receiver, the probability to be at that place. This is called "data fusion", and is investigated in the book [MPPR].

A simple example of such a situation is given by lines of sight: a light house is seen by 3 vessels, or more (or by a single vessel which moves); each vessel indicates its line of sight, but the lines do not converge at a single point.



A third type of application is the one we meet when partial information is given: for instance, we have indications about the level of pollution in a field, at some places (this is an artificially generated information) and we want to reconstruct the level of pollution at other places. Such problems are investigated in the book [PIT].

References

[AMW] Bernard Beauzamy : Archimedes' Modern Works. Société de Calcul Mathématique SA. ISBN 978-2-9521458-7-9, ISSN 1767-1175, August 2012.

[MPPR] Bernard Beauzamy : Méthodes Probabilistes pour l'étude des phénomènes réels. Société de Calcul Mathématique SA, ISBN 2-9521458-0-6, ISSN 1767-1175, March 2004.

[PIT] Olga Zeydina and Bernard Beauzamy: Probabilistic Information Transfer. Société de Calcul Mathématique SA. ISBN: 978-2-9521458-6-2, ISSN : 1767-1175. May 2013.